THERMOPARAMETRIC EXCITATION OF NONLINEAR VIBRATIONS OF A STRING

A. I. Potapov and V. V. Stupin UDC 534.112

Many thin-walled structural elements are compelled to function in transient temperature fields. As a result of heat transfer with the environment, fluctuating thermoelastic stresses are set up in the elements, where they can lead to dynamic loss of stability [1]. The simplest system in which such effects are observed is a metal string or fine wire with an alternating electric current flowing in it [2]. The alternating temperature field created by Joule heat induces a period modulation of the tension on the string and, under certain conditions, the parametric excitation of vibrations.

We have investigated experimentally and theoretically the excitation of transverse vibrations of a string (flexible wire) with a high-density alternating current flowing in it. The theoretical model adopted here can be used to calculate the thresholds of parametric instability and the dependence of the steady-state vibration amplitude on the ac power and the deviation from resonance, all with a high degree of accuracy. On the other hand, it does not predict the existence of hard excitation regimes or the cutoff of vibrations.

EXPERIMENTAL

When an alternating electric current at the line frequency (50 Hz) flows in a metal string (wire), its temperature rises as a result of Joule heat. In the experimental work, the static elongation of the wire due to the increase in its mean temperature was selected under the action of a tensile load N_0, which specified the constant component of the tension.

The experimental arrangement is shown schematically in Fig. 1, including the "string" of high-resistance wire 1, the load 2 governing the constant tension, and the tapped coil 3.

The current I_0 and the voltage across the working section were measured by an ammeter (A) and voltmeter (V) of precision class 1.5. The temperature of the wire was determined with a Promin' industrial pyrometer, which is capable of measuring the temperature in the range 1075-5273°K within 1-1.5% error limits. The tension N_0 on the string was set by means of a lever device and was varied discretely in 10^{-3} N steps by means of small weights. This technique permitted a reasonably smooth variation of the natural frequency of the string, which was calculated according to the equation

$$\omega_n = \frac{\pi n}{l} \sqrt{\frac{N_0}{\rho S} [1 + \alpha_T (T_m - T_0)]},$$

where n is the order of the vibrational mode, ρ is the density, S is the cross section of the string at the temperature T_0, α_T is the coefficient of linear expansion, and T_m, T_0 are the mean absolute temperatures of the heated string and the cold string, respectively. The angular frequency ω_n was monitored stroboscopically. Thermodynamic excitation of the first nine modes was observed in the experiments. The higher the mode order in this case, the softer was the excitation. In higher modes ($n \geqslant 4$), however, the vibration amplitude was small and therefore difficult to measure, while the excitation of the first mode required greater tension, resulting in partial cutoffs in the heating of the string. Consequently, the majority of the experiments were conducted in the second mode. Wires made from higher-resistance materials (Nichrome, German silver, Constantan, etc.) with a diameter d = 0.1-0.5 mm and a length of 1 mm or more were used for the experimental string.

Here we give the experimental results for Nichrome wire Kh20N80 of diameter d = 0.35 mm and length l = 1 m. Figure 2 shows the resonance curve $A = A(\xi)$, along with the experimental data and the dependence of the string vibration amplitude on the relative deviation from resonance $\xi = (\omega_n - \omega)/\omega$ for ac power values W_1 = 107 W and W_2 = 170 W (points 3 and 4, $\omega = 100\pi$).

Gorkii. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 142-146, September-October, 1985. Original article submitted June 15, 1984.

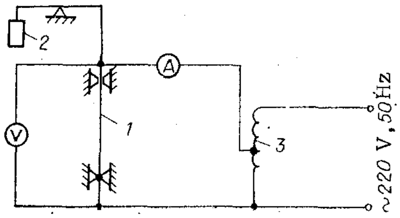

Fig. 1

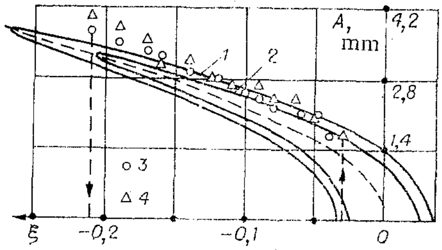

Fig. 2

The dependence of the steady-state vibration amplitude on the ac power is shown in Fig. 3 (the experimental data are represented by triangles). Figure 4 shows the mean temperature T_m as a function of the ac power W in the interval 1123–1430°K at the nodes (line 1) and antinodes (line 2) of the vibrating string. It is seen that the temperature of the string at the nodes is approximately 150–200°C higher than at the antinodes.

THEORETICAL ANALYSIS

The thermoparametric excitation of transverse vibrations of an elastic current-carrying string can be analyzed theoretically within the framework of nonlinear dynamical problems of thermoelasticity [1] with allowance for interaction between longitudinal and transverse vibrations of the string [3] and also for heat transfer between the heated string and the environment [4].

The following nonlinear boundary-value problem can be taken as the analytical model:

$$\frac{\partial^2 u}{\partial t^2} - v_l^2 \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] = -\frac{\alpha_T K}{\rho}\frac{\partial}{\partial x}(T - T_m); \quad (1)$$

$$\frac{\partial^2 w}{\partial t^2} - v_0^2\frac{\partial}{\partial x}\left\{\frac{\partial w}{\partial x}\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\} + \delta_1\frac{\partial w}{\partial t} + \delta_2 w^2\frac{\partial w}{\partial t} = 0; \quad (2)$$

$$\frac{\partial T}{\partial t} = \chi\frac{\partial^2 T}{\partial x^2} + \frac{2\alpha}{c\rho r}(T-T_0) + \frac{2\varepsilon\sigma}{c\rho r}(T^4 - T_0^4) = \frac{W(t)}{lSc\rho} - \frac{\alpha_T T_m K}{\rho c}\frac{\partial^2 u}{\partial x\,\partial t}; \quad (3)$$

$$w|_{x=0} = w|_{x=l} = 0,\quad u|_{x=0}=0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = \frac{N_0}{SE}, \quad (4)$$

where u(x, t) and w(x, t) are the longitudinal and transverse dispalcements of the string, respectively; $v_e = (e/\rho)^{1/2}$, $v_0 = (N_0/\rho)^{1/2}$ are the longitudinal and transverse wave velocities in the string; E and K are the Young's modulus and bulk compression modulus of the string material*; T is the absolute temperature of the string; T_m is the mean temperature of the heated string, $(T - T_m) \ll T_m$; $T_0 \approx 290$°K is the ambient temperature; ρ is the bulk density; δ_1 and δ_2 are the linear and nonlinear damping factors; $\chi = \varkappa/c$ is the thermal diffusivity; c is the specific heat of the material; α is the heat-transfer coefficient; r is the radius of the string, and $S = \pi r^2$ is its cross section; $\sigma = 5.6 \cdot 10^{-8}$ W/m$^2 \cdot$°K^4 is the Stefan–Boltzmann constant; $\varepsilon < 1$ is the emissitivity (graybody factor); and $W(t) = I_0 v_0 \cos^2 \omega t$ is the power of the Joule heat.

The first two equations describe coupled longitudinal–transverse vibrations of an elastic string [3] with allowance for thermal stresses [1], as well as the linear and nonlinear damping of transverse vibrations [5]. Equation (3) describes a thickness-uniform temperature field in the string in the presence of convective and radiative heat transfer with the surrounding air [4]. The terms on the right-hand side of (3) determine the heat release due to Joule heat and the self-heating due to the alternating longitudinal strains. The influence of the transverse vibrations of the string on its heat transfer with the surrounding air is ignored in the given equation. This problem has received very little attention either theoretically or experimentally. Only in occasional papers (see, e.g., [6]) is there an indication that transverse vibrations tend to increase the rate of convective heat transfer severalfold. We also neglect the relaxation processes mentioned in [7].

It appears to be impossible to solve the nonlinear problem (1)-(4) analytically in the general case. We introduce a number of simplifications, which follows from an analysis of the specific experimental situation. The small self-heating effect $\sim\partial^2 u/\partial x\partial t$ can be neglected

*The moduli E and K are evaluated at the mean temperature of the heated string $T = T_m$.

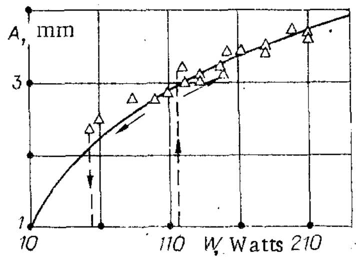

Fig. 3

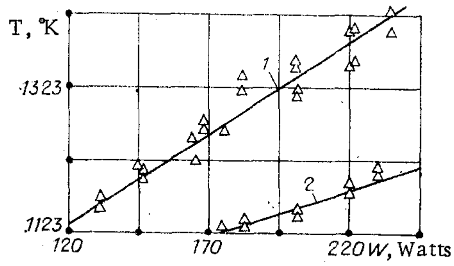

Fig. 4

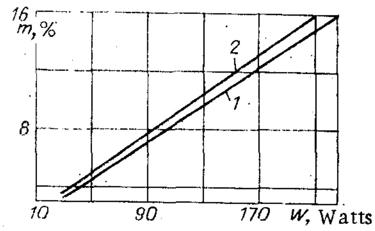

Fig. 5

in the heat-conduction equation in comparison with the Joule heat, and the heat transfer along the string $\sim\chi\partial^2T/\partial x^2$ can be neglected in comparison with the heat transfer to the surrounding air. Moreover, the velocity component $\partial u/\partial t$ is much smaller than $\partial w/\partial t$, and it can be neglected in Eq. (1) [3]. In this case the elongation of the string not depend on the coordinate x and it is a function of the time. Integrating Eq. (1) with respect to the coordinate and invoking the boundary conditions (4), we arrive at the problem

$$\frac{\partial^2 w}{\partial t^2}-v_0^2\left[1-\frac{\alpha_T K}{\rho v_0^2}\Theta(t)\right]\frac{\partial^2 w}{\partial x^2}-\frac{v_l^2}{2l}\left\{\int_0^l\left(\frac{\partial w}{\partial x}\right)^2dx\right\}\frac{\partial^2 w}{\partial x^2}+\delta_1\frac{\partial w}{\partial t}+\delta_2\frac{\partial w}{\partial t}w^2=0; \tag{5}$$

$$\frac{d\Theta}{dt}+\alpha_э\Theta=\frac{W_0}{lS\rho c}\cos 2\omega t; \tag{6}$$

$$\alpha(T_m-T_0)+\sigma\varepsilon\left(T_m^4-T_0^4\right)=\frac{W_0}{\pi rl}; \tag{7}$$

$$w|_{x=0}=w|_{x=l}=0, \tag{8}$$

where $\Theta(t) = T - T_m$ is the temperature fluctuation about the mean value T_m; $\alpha_e = 2(\alpha + 4\varepsilon\sigma T_m^3)/c\rho r$ is the effective heat-transfer coefficient; and $W_0 = (1/2)I_0v_0$.

Equation (7) characterizes the dependence of the mean temperature T_m of the string on the Joule heat power, and Eq. (6) describes its small time fluctuations:

$$\Theta(t)=\Theta_0\sin(2\omega+\psi_0),\quad \Theta_0=\frac{W_0}{Sl\rho c\sqrt{\alpha_e^2+4\omega^2}},\quad \operatorname{tg}\psi_0=\frac{\alpha_e}{2\omega}.$$

For a known law governing the variation of the temperature $\Theta(t)$, the integrodifferential equation (5) describes the nonlinear thermoparametric vibrations of the string. In the experiments, as a rule, we observed the excitation of vibrations at one of the natural frequencies of the system, and so we confine the discussion to particular solutions of the form

$$w_n(x,t)=q_n(t)\sin k_n x,$$

where $k_n = (\pi n/l)$ $(n = 1, 2, 3,\ldots)$ is the wave number of the excited mode. For $q_n(t)$ we obtain from (5) a nonlinear equation that is well known in the theory of parametric vibrations [5, 8, 9]:

$$\ddot q_n+\omega_n^2[1+m\sin(2\omega+\psi_0)]q_n+\frac{v_l^2k_n^4}{4}q_n^3+\left(\delta_1+\frac{3}{4}\delta_2q_n^2\right)\dot q=0, \tag{9}$$

where $m=\alpha_T k\Theta_0/\rho v_0^2$ is the modulation factor for the tension of the string and $\omega_n = v_0k_n$ is the natural frequency of the string in the linear approximation. The solution for the principal domain of parametric instability is sought in the form

$$q_n(t)=A(\mu t)\sin(\omega t+\varphi(\mu t))+\mu Q(t), \tag{10}$$

where $A(\mu t)$, $\varphi(\mu t)$ are the slowly varying amplitude and phase of the vibrations and $\mu Q(t)$ is a small correction to the solution, $\mu \ll 1$. Requiring boundedness of $Q(t)$, we obtain the following truncated equations for A and φ by the averaging method in [9]:

$$\begin{aligned}\dot A_n&=-\frac{m}{2}\frac{\omega_n^2}{\omega}A_n\cos 2\varphi_n-\left(\delta_1+\frac{3}{16}\delta_2A_n^2\right)A_n,\\ \dot\varphi_n&=\frac{\omega_n^2-\omega^2}{\omega}+\frac{m}{2}\frac{\omega_n^2}{\omega}\sin 2\varphi_n+\frac{3}{16}\frac{\beta k_n^4A_n^2}{\omega}.\end{aligned} \tag{11}$$

The equilibrium state $\dot A=0$, $\dot\varphi=0$ of the truncated system of equations (11) corresponds to steady-state periodic vibrations of the string. The dependence of the steady-state vibration amplitude on the frequency deviation has the form

$$\xi = -\frac{3}{32}\frac{\beta k_n^4}{\omega^2}A_n^2 \pm \frac{1}{2}\sqrt{\frac{m^2}{4}\left(\frac{\omega_n}{\omega}\right)^4 - \frac{1}{\omega^2}\left(\delta_1 + \frac{3}{16}\delta_2 A_n^2\right)^2}. \qquad (12)$$

Here we have introduced the notation $\xi \approx (\omega_n - \omega)/\omega \approx (\omega_n^2 - \omega^2)/2\omega^2$, which characterizes the relative deviation of the current fluctuation frequency ω from the small-amplitude natural frequency ω_n of the string. Expression (12) describes resonance curves emanating from the points

$$\xi_{max} = \pm\frac{1}{4}\left(\frac{\omega_n}{\omega}\right)^2\sqrt{m^2 - \left(\frac{\delta_1}{\omega}\right)^2},$$

which merge at the point corresponding to the maximum vibration amplitude:

$$A_{max} = \sqrt{\frac{8\left(m\frac{\omega_n^2}{\omega} - 2\delta_1\right)}{3\delta_2}}. \qquad (13)$$

From this result we readily deduce the parametric instability threshold ($\xi = 0$, $A_n = 0$):

$$m_* = \frac{2\delta_1}{\omega_n}\left(\frac{\omega}{\omega_n}\right). \qquad (14)$$

We note that whereas the vibration amplitude (12) exhibits a complicated dependence on all the quantities involved in Eq. (9), the instability threshold m_* and the maximum vibration amplitude A_{max} are determined only by the modulation factor m for the tension of the string and by the linear and nonlinear damping factors δ_1 and δ_2. The nonlinear restoring force governs only the frequency deviation (skeleton curve). The stability problems of the periodic solutions (10) have been treated in detail previously [8, 9] and will not be discussed here.

DISCUSSION

The results can be used to compare the experimental data with theoretical calculations based on the model (5)-(8). Using the experimental data, we can calculate the heat-transfer coefficient α according to (7). It varies linearly from 50 to 60 $W/m^2\cdot°K$ in the temperature interval 500-1500°K, and it can be considered practically constant on the working section. Knowing all the necessary constants of the material† (Nichrome Kh20N80) [10, 11], we can find the relative temperature fluctuation Θ_0/T_m and the tension modulation factor m. The calculations show that the tension factor m varies by tens of percents, even though the temperature of the string varies by less than 0.05% of its mean value as a result of transient heat transfer (Fig. 5, in which lines 1 and 2 correspond to nodes and antinodes, respectively). This amount is sufficient for the parametric excitation of vibrations, as was indeed observed experimentally. Relations (13) and (14) enable us to calculate the damping factors, which are equal to the following in the given situation: $\delta_1 = 1.5\ sec^{-1}$; $\delta_2 = 1.5\cdot10^7\ sec^{-1}\cdot m^{-2}$. With the use of these constants, relations (12) permit the amplitude of the steady-state vibrations to the plotted as a function of the frequency deviation (see Fig. 2, in which curves 1 and 2 correspond to m = 0.12 and m = 0.14, respectively). Expression (13) describes the dependence of the steady-state amplitude on the ac power at a constant deviation ξ_0 (see the solid curve in Fig. 3; $\delta_1 = 1.5\ sec^{-1}$, $\delta_2 = 1.5\cdot10^7\ sec^{-1}\cdot m^{-2}$, $\xi_0 \approx -0.20$).

Comparing the theoretical calculations with the experimental data, we arrive at the following conclusion. The adopted theoretical model correctly predicts the possibility of thermoparametric excitation of vibrations in a string carrying an alternating electric current. It permits the parametric instability threshold to be calculated quite accurately, along with the dependence of the steady-state vibration amplitude on the ac power and the deviation of the temperature fluctuation frequency from the natural frequency of the string (ξ). On the other hand, it cannot be used to predict the existence of such an essentially nonlinear effect as the hard excitation regime and the cutoff of vibrations. The latter result is evidently associated with the existence of singular aspects of the heat transfer process between the vibrating string and the surrounding air that are not described by the heat-conducting equation (3). The detailed discussion of this topic poses a complex problem and is beyond the scope of the present study.

†In the calculations, allowance has been made for the fact that the Young's modulus E decreases by approximately 20-25% according to a linear law with increasing temperature in the working range.

The authors are indebted to A. I. Vesnitskii for attention and interest, and also to N. D. Romanov for assistance with the experimental work.

LITERATURE CITED

1. N. I. Bezukhov, V. L. Bazhanov, et al., Strength, Stability, and Vibration Calculations under High-Temperature Conditions [in Russian], Mashinostroenie, Moscow (1965).
2. V. I. Iveronova, A. G. Belyankin, et al., Practical Work in Physics [in Russian], Nauka, Moscow (1967).
3. H. Kauderer, Nonlinear Mechanics [Russian translation], IL, Moscow (1961).
4. S. S. Kutateladze, Fundamentals of Heat-Transfer Theory [in Russian], Atomizdat, Moscow (1979).
5. V. V. Bolotin, Dynamical Stability of Elastic Systems [in Russian], Gostekhizdat, Moscow (1956).
6. F. Ruszel and W. Tomezak, "Heat transfer from cylinder to liquid with applied vibration," Arch. Termodyn., 3, No. 1 (1982).
7. G. A. Gemmerling, "Dependence of the internal energy of a solid on the rate of change of temperature," Dokl. Akad. Nauk SSSR, 180, No. 5 (1968).
8. G. Schmidt, Parametric Vibrations [Russian translation], Mir, Moscow (1978).
9. P. N. Bogolyubov and Yu. A. Mitropol'skii, Asymptotic Methods in the Theory of Nonlinear Vibrations [in Russian], Nauka, Moscow (1974).
10. G. I. Pogodin-Alekseev (ed.), Handbook of Mechanical Engineering Materials. Vo;. 2: Nonferrous Metals and Their Alloys [in Russian], M. A. Bochvar (ed.), Mashgiz, Moscow (1959).
11. B. G. Lifshits, V. S. Kraposhin, and L. A. Lipetskii, Physical Properties of Metals and Alloys [in Russian], Metallurgiya, Moscow (1980).

DETERMINATION OF THE EFFECTIVE ELASTIC MODULI OF INHOMOGENEOUS MATERIALS

V. V. Novikov

UDC 539.3

1. FORMULATION OF THE PROBLEM

Quasihomogeneous media that possess effective properties dependent on the properties, volume concentration, and contact conditions of the components are usually investigated when examining the effective properties of inhomogeneous materials. The necessary and sufficient condition for going over to the quasihomogeneous medium is compliance of the dimension of the inhomogeneity l with the inequality

$$l_0 \ll l \ll L, \tag{1.1}$$

where l_0 is the crystal lattice constant and L is the specimen dimension.

The effective elastic moduli C_{ijkl} and the pliability S_{ijkl} are determined from the equations

$$\langle\sigma_{ij}\rangle = C_{ijkl}\,\langle\varepsilon_{kl}\rangle,\ \langle\varepsilon_{ij}\rangle = S_{ijkl}\ \langle\sigma_{kl}\rangle. \tag{1.2}$$

The angular brackets <...> here denote taking the average over the volume of the material

$$\langle\sigma_{ij}\rangle = \frac{1}{V}\iiint\limits_V \sigma_{ij}(\mathbf{r})\,dx_1dx_2dx_3,\quad \langle\varepsilon_{ij}\rangle = \frac{1}{V}\iiint\limits_V \varepsilon_{ij}(\mathbf{r})\,dr_1dr_2dr_3. \tag{1.3}$$

The equations

$$\sigma_{ij}(\mathbf{r}) = C_{ijkl}(\mathbf{r})\varepsilon_{kl}(\mathbf{r}),\ \varepsilon_{ij}(\mathbf{r}) = S_{ijkl}(\mathbf{r})\sigma_{kl}(\mathbf{r}), \tag{1.4}$$

are valid for the local domains (components) when conditions (1.1) are satisfied, where $\sigma_{ij}(\mathbf{r})$ is the local stress tensor, $\varepsilon_{ij}(\mathbf{r})$ is the local strain tensor, and $\mathbf{r} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is a radius-vector.

Vologda. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 146-153, September-October, 1985. Original article submitted June 14, 1984.

